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LATTICE NONEMBEDDINGS AND INITIAL SEGMENTS OF THE RECURSIVELY ENUMERABLE DEGREES

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1. Introduction

This paper studies the structure of a general initial segment $[\mathbf{0}, \mathbf{a}]$ $(\mathbf{a} \neq \mathbf{0})$ of the upper semilattice $\mathbf{R}$ of r.e. degrees. In particular we address the question of what lattices may be embedded into all such segments. It turns out that some $[\mathbf{0}, \mathbf{a}]$ may be very different from $\mathbf{R}$ since many lattices (and semilattices) embeddable into $\mathbf{R}$ are not embeddable into such $[\mathbf{0}, \mathbf{a}]$. The general theme seems to be that for $\mathbf{a}$ 'sufficiently close to $\mathbf{0}'$', $[\mathbf{0}, \mathbf{a}]$ is 'much more' distributive than is $\mathbf{R}$. For example, we can show

(1.1) **Theorem.** *A countable modular lattice L is embeddable into all nontrivial initial segments of $\mathbf{R}$ iff L is distributive.*

Here the reader should recall that a lattice L is distributive if for all $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and that a lattice L is called modular if whenever $x \leq y$ then $x \vee (z \wedge y) = (x \vee z) \wedge y$. All distributive lattices are modular. Furthermore the following elementary results are relevant here: a lattice L is modular iff the lattice N_5 of Diagram 0 is not embeddable into L; and L is distributive iff both N_5 and M_5 of Diagram 0 are not embeddable into L.

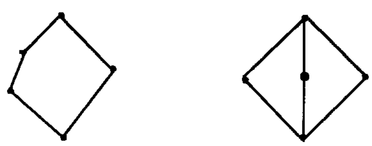

Diagram 0

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Part of the proof of Theorem (1.1) is a straightforward embedding result of Ambos-Spies [1] which we give in Section 2.

(1.2) **Theorem** (Ambos-Spies [1]). *All countable distributive lattices are embeddable into all nontrivial initial segments of* $\mathbf{R}$.

The difficult half of Theorem (1.1) is a very strong nonembedding result we prove in Section 3. This result actually gives quite a bit more. We need the following definition.

(1.3) **Definition.** Let $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2 \in \mathbf{R}$. We say $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ form a *critical triple* if $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{a}_1 \cup \mathbf{a}_2$, $\mathbf{a}_0 \not\leq \mathbf{a}_1$ and $\forall \mathbf{c}\,(\mathbf{c} \leq \mathbf{a}_0, \mathbf{a}_2 \rightarrow \mathbf{c} \leq \mathbf{a}_1)$.

For example, in Diagram 1, critical triples are identified in some typical lattices.

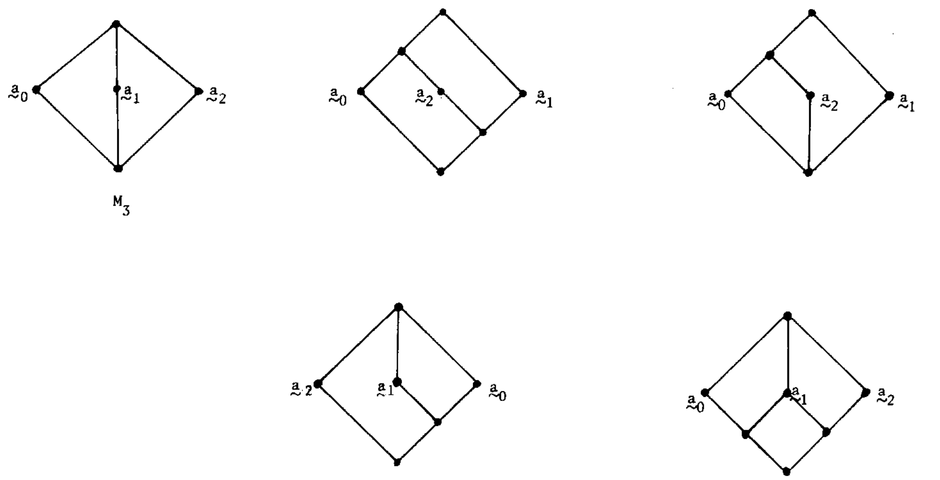

Diagram 1

The main result of the paper is

(1.4) **Theorem.** *There exists an r.e. degree* $\mathbf{a} \neq \mathbf{0}$ *that bounds no critical triple. Indeed each nonzero r.e. degree has a predecessor with this property.*

Of course, Theorem (1.1) follows from (1.2) and (1.4) since M_3 is embeddable in all modular nondistributive lattices. At this stage we would like to put on record the fact that we feel the converse of (1.4) holds, viz:

(1.5) **Conjecture.** *If L is a lattice that contains no critical triple, then L is embeddable below any nonzero r.e. degree.*

It may even be possible to show that if L is such a lattice, then L embeds into $[\mathbf{a}, \mathbf{b}]$ for any $\mathbf{a} < \mathbf{b}$. It does seem quite probable that the density version of (1.1) will hold. We have proven the analogue of (1.2) in [4] extending an earlier result of Slaman [12]. Hence the analogue of Theorem (1.1) follows: *A countable modular lattice L is embeddable into all nontrivial intervals of* $\mathbf{R}$ *iff L is distributive.*

We point out that all of the lattices in our examples above *are* embeddable into $\mathbf{R}$ (Lachlan [9], Ambos-Spies and Lerman [2]). We also feel that our results indicate that the highly nondistributive techniques of Harrington–Shelah–Slaman [7, 8] are unlikely to work for arbitrary $[\mathbf{0}, \mathbf{a}]$.

Notation is standard and can be found in Soare [13]. We also follow the convention that he uses, where defined are monotone in argument and stage number. As usual all computations, etc. are bounded by $s-1$ at stage s. Both of our arguments use 'tree of strategy' arguments. The argument of Section 3 uses the $\mathbf{0}'''$-method presented in the manner of Slaman/Soare in Soare [13]. It is useful, but not essential if the reader is familiar with that account.

2. Embeddings

We establish (1.2) by proving

(2.1) **Theorem.** *The (countable) atomless boolean algebra* $\mathbf{Q}$ *embeds into any nontrivial initial segment of* $\mathbf{R}$.

Proof. Actually, our proof involves only a variation on the classical Lachlan–Lerman–Thomason embedding of $\mathbf{Q}$ into $\mathbf{R}$ preserving $\mathbf{0}$. Thus we don't really discuss the strategies in details; rather we give the construction and refer the reader to Soare [13, Ch. IX] for further motivation. Let E be a given r.e. nonrecursive set with canonical enumeration $\bigcup_s E_s$.

Let $\{\alpha_i\}_{i\in\omega}$ be any uniformly recursive sequence of recursive sets forming $\mathbf{Q}$ under $\cup$, $\cap$ and $-$, contains ω and has $\emptyset$ as its only finite member. We will construct r.e. sets $A_i = \bigcup_s A_{i,s} \leq_T E$ and define

$$A_\alpha = \{\langle i, x\rangle : x \in A_i \text{ and } i \in \alpha\} \quad \text{for } \alpha \in \mathbf{Q}.$$

Here α will sometimes denote an infinite set and sometimes a corresponding index.

This well-known representation trick gives $\deg(A_{\alpha\cup\beta}) = \deg(A_\alpha) \cup \deg(A_\beta)$,

$$\alpha \subset \beta \ \Rightarrow\ \deg(A_\alpha) \leq \deg(A_\beta) \quad \text{and} \quad \deg(A_{\alpha\cap\beta}) \leq \deg(A_\alpha),\ \deg(A_\beta).$$

We build an additional set $Q \leq_T E$ and for all e, α, β meet the requirements

$$P_{\langle e,i\rangle}:\quad \Phi_e(Q) \neq A_i,$$

$$N_{\langle \alpha,\beta,e\rangle}:\quad \Phi_e(A_\alpha \oplus Q) = \Phi_e(A_\beta \oplus Q) = f \text{ total} \Rightarrow f \text{ recursive in } A_{\alpha\cap\beta} \oplus Q.$$

As usual (see e.g. [13, Ch. IX, §2]) this guarantees $\alpha \to \deg(A_\alpha)$ is the desired isomorphism. For $e = \langle \alpha, \beta, i \rangle$, let

$$l(e, s) = \max\{x : (\forall y < x)[\Phi_{i,s}(A_{\alpha,s} \oplus Q_s; y) = \Phi_{i,s}(A_{\beta,s} \oplus Q_s; y)]\}.$$

As in the (tree version of the) minimal pair type argument of Lachlan–Lerman–Thomason, define the notion of *σ-stage* by induction on $\mathrm{lh}(\sigma)$ for $\sigma \in 2^{<\omega}$ via

(i) Every stage s is a λ-stage (where λ denotes the empty string).

(ii) If s is a τ-stage and $\mathrm{lh}(\tau) = \langle \alpha, \beta, i \rangle = e$, then if $l(e, s) > \max\{t + 1 : t$ is a $\tau\hat{}0$-stage and $t < s\}$, then s is a $\tau\hat{}0$-stage. (Here 0 is a $\tau\hat{}0$-stage.) Otherwise s is a $\tau\hat{}1$-stage.

We let σ_s denote the unique string γ such that $\mathrm{lh}(\gamma) = s$ and s is a γ-stage. As usual let $\leqslant_{\mathrm{L}}$ denote lexicographic ordering with $0 <_{\mathrm{L}} 1$.

Now let

$$L(e, i, s) = \max\{x : (\forall y < x)(\Phi_{e,s}(Q_s; y) = A_{i,s}(y))\}.$$

We attempt to meet the requirements $P_{\langle e,i \rangle}$ by followers y. These will be of the form $y(\sigma, x, s)$ to indicate they have guess σ (where $\mathrm{lh}(\sigma) = \langle e, i \rangle$) and have *permitting number* x. Such a follower can only be appointed at σ-stages when all smaller followers are 'realized' (i.e. ready to be permitted). The definition of realized is deferred until later.

Once realized $y(\sigma, x, s)$ will enter $A_{i,s}$ only if E permits x at s. The difference between this construction and a minimal pair type of construction is that we must be able to enumerate y whenever E so permits, whereas in (e.g.) a minimal pair argument we must await a σ-stage to so enumerate x. As we know (Lachlan [10]) such waiting is a strong enough obstacle that in fact not all r.e. degrees bound minimal pairs. The additional trick we shall use will be a *Q-marker* $q(y, s)$ *tied to* y which we shall use to allow Q to recover 'both sides' of a changed computation. (Similar ideas were used in [3].)

The reader should note that in our construction to follow the definition of $\sigma\hat{}0$-stage above has the following consequence. Suppose y is a follower appointed at a $\sigma\hat{}0$-stage s. Suppose $\hat{s}$ is a $\sigma\hat{}0$-stage larger than s and $\tau\hat{}0 \subset \sigma\hat{}0$ with $\mathrm{lh}(\tau) = e$. Then $l(e, \hat{s}) > y$. This will follow as we can only appoint s as a follower at stage s as we will see. Now we give the formal details.

We say that $P_{\langle e,i \rangle}$ *requires attention* at stage $s + 1$ if one of the following options holds.

(2.2) There is a follower $y = y(x, \sigma, s)$ (say) of $P_{\langle e,i \rangle}$ with $\mathrm{lh}(\sigma) = \langle e, i \rangle$ for some $\sigma \leqslant_{\mathrm{L}} \sigma_s$ such that

(i) E permits x at s, and

(ii) y is realized at stage s.

(2.3) $P_{\langle e,i \rangle}$ has an unrealized follower $y = y(x, \sigma, s)$ with $\sigma \subset \sigma_s$ such that $L(e, i, s) > y$ where $\mathrm{lh}(\sigma) = \langle e, i \rangle$ and (2.2) does not hold.

(2.4) Case (2.2) does not pertain and $P_{\langle e,i\rangle}$ has no follower $y(0, \sigma, s)$ for $\sigma \subset \sigma_s$ and $\mathrm{lh}(\sigma) = \langle e, i\rangle$.

Construction, stage $s+1$

Step 1. Cancel all $y = y(x, \tau, s)$ and $q(y, s)$ for $\tau \not\leq_{\mathrm{L}} \sigma_s$.

Step 2. Find the least $\sigma \leq_{\mathrm{L}} \sigma_s$ such that for $\langle e, i\rangle = \mathrm{lh}(\sigma)$, $P_{\langle e,i\rangle}$ requires attention via some (least) $y(\sigma, x, s)$ or $\sigma \subseteq \sigma_s$ and (2.4) pertains. Adopt the first case below to hold.

Case 1: (2.2) *holds*. Set $A_{i,s+1} = A_{i,s} \cup \{y\}$, $Q_{s+1} = Q_s \cup \{q(y)\}$ and cancel all numbers $\geq y$ and all followers of the form $y(\hat{z}, \sigma, s)$ together with their Q-markers.

Case 2: (2.3) *holds*. Declare y as realized. Cancel all numbers $>y$. Appoint $y(x+1, \sigma, s+1) = s+1$ as a new follower of $P_{\langle e,i\rangle}$ for guess σ and set $q(y(x, \sigma, s), s+1) = s+1$ too.

Case 3: (2.4) *holds*. Appoint $y(0, \sigma, s) = s+1$ as an unrealized follower of $P_{\langle e,i\rangle}$ at guess σ and initialize all $\tau \not\leq_{\mathrm{L}} \sigma$. □ End of Construction

Verification. The proof that each $P_{\langle e,i\rangle}$ receives attention finitely often along the true path is rather straightforward so we only sketch the details. Let γ denote the true path (i.e. $\gamma \in [2^{<\omega}]$ and γ is defined inductively via $\lambda \subset \gamma$ and if $\tau \subset \gamma$, then $\tau^\frown 0$ iff there are infinitely many $\tau^\frown 0$-stages; otherwise $\tau^\frown 1 \subset \gamma$). Let $\sigma \subset \gamma$ and suppose $\mathrm{lh}(\sigma) = \langle e, i\rangle$. For an induction suppose s_0 is a stage good for σ in the sense that for stages t after stage s_0, $\sigma \leq_{\mathrm{L}} \sigma_t$ and for all $\hat{\sigma} \subsetneq \sigma$, no followers with guess $\hat{\sigma}$ act or are appointed.

By initialization—when some $P_{\langle \hat{e}, \hat{i}\rangle}$ receives attention—we can suppose $P_{\langle e,i\rangle}$ has no followers with guess σ at stage s_0. Now at the next σ-stage after stage s_0, $P_{\langle e,i\rangle}$ will get a follower $y_0 = y(0, \sigma, s)$ which will be uncancellable. Clearly $P_{\langle e,i\rangle}$ is now met unless (2.3) pertains to y_0. Now when this occurs—say at stage s_0—we cancel all followers $>y_0$ and set $y_1 = q(y_0) = s_0$. By our conventions note that $y_1 = q(y_0) = s_0$ exceeds the use of $\Phi_{e,s_0}(Q_{s_0}; y_0) = A_{i,s_0}(y_0)$. It is easy to see that in general we get a potentially infinite recursive sequence of uncancellable followers

$$y_0, \quad y_1 = q(y_0), \quad y_2 = q(y_1), \ldots$$

Each of these has the property that $q(y_{k+1}) > u(\Phi_{e,s_{k+1}}(Q_{s_{k+1}}; y_k))$. By a standard permitting argument, as E is nonrecursive, E must permit some k above. Suppose this occurs at stage $t > s_{k+1}$. Then by our cancellation procedure and since the way we appoint followers as stage numbers, we know

$$\Phi_{e,t}(Q_t; y_k) = \Phi_e(Q; y_k) = 0 \neq 1 = A_i(y_k).$$

Note that at the least σ-stage $t_1 > t$, $P_{\langle e,i\rangle}$ will get an *unrealizable* follower t_1 and so $P_{\langle e,i\rangle}$ will be met and never again receive attention.

Now we argue that all the N_e are met. Let $\sigma \subset \gamma$ with $\mathrm{lh}(\sigma) = e = \langle \alpha, \beta, i\rangle$. Let s_0 be a stage good for σ as above. To see that N_e is met we suppose

$\Phi_i(A_\alpha \oplus Q) = \Phi_i(B_\alpha \oplus Q) = f$ total, and so $\sigma\hat{}0 \subset \gamma$. Let z be given. To compute $f(z)$ find the least $\sigma\hat{}0$-stage $t > s_0$ such that $l(e, t) > z$. Now compute the least $\sigma\hat{}0$-stage $v > t$ such that

$$(2.5)\qquad Q_v[t] = Q[t] \quad\text{and}\quad A_{\alpha\cap\beta,v}[t] = A_{\alpha\cap\beta}[t].$$

We claim that $f(z) = f_v(z)$. To see this, we show by induction that for all $s > v$

$$(2.6)\qquad \text{one of}\quad \Phi_{i,s}(A_{\alpha,s} \oplus Q_s; z) = \Phi_{i,v}(A_{\alpha,v} \oplus Q_v; z) \quad\text{or}$$
$$\Phi_{i,s}(A_{\beta,s} \oplus Q_s; z) = \Phi_{i,v}(A_{\beta,v} \oplus Q; z) \quad\text{holds.}$$

If (2.6) is to fail, then at some least $\sigma\hat{}0$-stages $s_1 > s_2 \geq v$—with s_2 the preceding $\sigma\hat{}0$-stage before s_1—there must be two numbers y and $\hat{y}$ which enter respectively the A_α-side and the A_β-side below the s_2 uses at stages r_1 and r_2 with $s_2 \leq r_1$, $r_2 < s_1$ respectively. The reason we can take s_1 and s_2 to be $\sigma\hat{}0$-stages is that—as with a minimal pair argument—if only one side changes between $\sigma\hat{}0$-stages, then as the computations of *both* sides hold at $\sigma\hat{}0$-stages, we must get (2.6). We shall argue that this is impossible.

First we claim that in fact two numbers must enter as *followers* between stages s_1 and s_2 (and so not as Q-markers). To see that this is the case, if a number is enumerated as a Q-marker it must enter Q.

We claim that if q is a Q-marker with q entering Q_n and $q \leq u(\Phi_{i,s}(A_{\alpha,s} \oplus Q_s; z))$ at any stage $n > s \geq t$ for any $\sigma\hat{}0$-stage s, then q was *already appointed* at stage t. This will then contradict (2.5).

The point is that if q is appointed at or after stage t, then q is appointed at a $\sigma\hat{}0$-*stage* $t_1 \leq s$ after stage t. (Else it would die at s.) Thus, by convention and definition of $\sigma\hat{}0$-stage, $q = t_1$ and exceeds both uses. It is easy to see that when q is appointed then for $q = q(\hat{y})$, $\hat{y}$ is the largest follower defined at t_1. Now if q enters it can only be *at the same stage* n *as* $\hat{y}$. No follower $\leq t_1$ can have entered A_α or B_α at any stage t_2 with $t_1 \leq t_2 \leq n$ lest it cancel $\hat{y}$ and q. Thus as q is still alive we see that q still exceeds both uses since the computations are unchanged since stage t_1. The claim then follows.

Thus we may take y and $\hat{y}$ to be followers. Note that by the same argument as above (cancellation and appointment at $\sigma\hat{}0$-stages) we can see that both y and $\hat{y}$ must have been already appointed at stage t. Note that our assumption (2.5) on $A_{\alpha\cap\beta}$ means that $y \neq \hat{y}$ and so without loss of generality $y < \hat{y}$. Now both y and $\hat{y}$ must be realized when they enter, as only realized followers enter A_α or A_β. Realization can only occur at $\sigma\hat{}0$-stages and hence both must be realized at stage s_2. Now since the realization of y at stage r would have cancelled all followers $g > y$ defined at stage r, it must have been the case that y was realized at stage t (since $\hat{y}$ is alive). It therefore follows (and this is the whole point) that $q(y) \leq t$ and hence by (2.5), $q(y) \in Q$ iff $q(y) \in Q_v$. Thus y cannot enter A_α or A_β after stage v after all, and the claim (2.6) follows, concluding the proof of (2.1). $\square$

3. The main result

(3.1) **Theorem.** *There exists an r.e. degree* $\mathbf{a} \neq \mathbf{0}$ *that bounds no critical triple.*

Proof. We build $A = \bigcup_s A_s$ in stages, together with auxiliary r.e. sets $Q_e = \bigcup_s Q_{e,s}$ to meet the requirements

P_e: $\bar{A} \neq W_e$,

R_e: For $j = 0, 1, 2$, and $k = 0, 2$, if $\Lambda_e^j(A) = W_e^j$ and $\Phi_e^k(W_e^{k+1} \oplus W_e^{k+2}) = W_e^k$ (of course, mod 3) then either $W_e^0 \leqslant_{\mathrm{T}} W_e^1$ or $Q_e \leqslant_{\mathrm{T}} W_e^0, W_e^2$ and $\forall j\,(\Psi_j(W_e^1) \neq Q_e)$.

Here $\{\Psi_e\}_{e \in \omega}$ is a list of all functionals and $\langle \Lambda_e^0, \Lambda_e^1, \Lambda_e^2, \Phi_e^0, \Phi_e^2, W_e^0, W_e^1, W_e^2 \rangle$ is a list of all 8-tuples consisting of five functionals and three r.e. sets.

We shall need some auxiliary functions.

$$\lambda^i(e, s) = \max\{x : (\forall y < x)[\Lambda_{e,s}^i(A_s; y) = W_{e,s}^i(y)]\} \quad \text{for } i = 0, 1, 2,$$

$$L(e, i, s) = \max\{x : (\forall y < x)[\Psi_{i,s}(W_{e,s}^1(y)) = Q_{e,s}(y) \\ \& \; l(e, s) > u(\Psi_{e,s}(W_{e,s}^1(y)))]\}.$$

and finally the A-controllable length of agreement

$$l(e, s) = \max\{x : (\forall z < x)(\forall k \in \{0, 2\})[\Phi_{e,s}^k(W_{e,s}^{k+1} \oplus W_{e,s}^{k+2}; z)] = W_{e,s}^k(z) \\ \& \; (\forall y)[y \leqslant u(\Phi_{e,s}^k(W_{e,s}^{k+1} \oplus W_{e,s}^{k+2}; z)) \rightarrow \\ (\forall j \leqslant 2)[\lambda^j(e, s) > y]]\}.$$

For $l(e, s) > x$ we similarly define the total use function $u(x, e, s)$ so that if we preserve $A_s[u(x, e, s)]$, then we hold $l(e, s) > x$ with the computations unchanged. Here we apply the usual conventions: if $\Omega_s(B_s; y) = W_{e,s}(y)$ and we preserve $B_s[u(\Omega_s(B_s; y))]$, then we don't allow new numbers to enter $W_{e,s}[y]$. Of course in our construction this does no harm since we are only concerned with those Ω, B and W_e for which $\Omega(B) = W_e$. Also, we presume that—where defined—use functions are increasing both in argument and stage number, (if reset). The last convention saves considerably on notation.

We shall meet P_e by a standard Friedberg argument (on a tree) and don't need to discuss this in detail. The key requirements are of course the R_e. Before we give the details of the basic module we shall discuss the motivation for some of the new technical devices (such as *layering*) used in our construction. It is hoped that the reader can also better glean the basic 'shape' of the general construction by initially stripping away some of the more formal details.

We first break R_e into infinitely many subrequirements, each to be implemented in the full construction as π_2 guessing nodes. The easiest subrequirements are

$\hat{R}_e$: for some $j \in \{0, 1, 2\}$ or $k \in \{0, 2\}$ either

$\Lambda_e^j(A) \neq W_e^j$ or $\Phi_e^k(W_e^{k+1} \oplus W_e^{k+2}) \neq W_e^k$.

To test $\hat{R}_e$ we need to see if $l(e, s) \to \infty$. We remark that as in the Slaman/Soare account of the $\mathbf{0}'''$-method (Soare [13]), a node τ devoted to $\hat{R}_e$ will be the top of 'links' to the subrequirements $R_{e,i}$ below.

If $\hat{R}_e$ fails to be met, we must build $Q_e \leq_T W_e^0$, W_e^2 and attempt to meet for all $i \in \omega$ the subrequirements

$$R_{e,i}: \quad \Psi_i(W_e^1) \neq Q_e.$$

Finally if for some least i, we fail to meet $\hat{R}_e$ and $R_{e,i}$ we must ensure that $W_e^0 \leq_T W_e^1$. This will be the outcome if $R_{e,i}$ receives attention infinitely often, and we will say $R_{e,i}$ has outcome g.

The rough idea is this. Assume $l(e, s) \to \infty$. We aim to ensure $\Psi_i(W_e^1) \neq Q_e$ by followers $x = x(e, i, s)$ and $Q_e \leq_T W_e^0$, W_e^2 by 'permitting'. Strictly speaking permitting is certainly not accurate. But it will do as a first approximation which we will later modify. The actual reductions $\Gamma^0(W_e^0) = W_e$ and $\Gamma^2(W_e^2) = Q$ will have uses $\gamma^0(x, e, s)$ and $\gamma^2(x, e, s)$ respectively. Our first attempt is to pick a follower x targeted for Q_e and wait till we see $l(e, s) > x$. (We call this a *confirmation* stage.) When this occurs we cancel all followers y targeted for A with $y > x$ (the reason for this becomes clear later) and restrain A on $u(x, e, s)$ preserving the current e-computations. Our first 'permitting' attempt is to set $\gamma^0(x, e, s) = x$ and $\gamma^2(x, e, s) = \lambda^2(e, s)$, and ask that x be allowed to enter Q_e only if both $W_e^0[x]$ and $W_e^2[\lambda^2(e, s)]$ change between *e-expansionary* stages (i.e. where $l(e, s) > ml(e, s) = \max\{l(e, t) : t < s\}$).

Now, we don't drop the restraint $r(e, i, s)$ on $A[u(x, e, s)]$ until we see a stage $s_1 \geq s$ where $L(e, i, s_1) > x$. Clearly, should s_1 not occur, we win (since $\Psi_e(W_e^1; y) \neq Q_e(y)$ for some $y \leq x$) with finite effect. Should we see s_1 occur, we then *open an* (e, i)*-gap* by setting $r(e, i, s_1) = 0$, potentially allowing $A[u(x, e, s)]$ to change. Our main hope is that when we close our (e, i)-gap, nice enough conditions will have occurred to allow us to enumerate x into Q_e in such a way as to create a preservable disagreement at x. As $Q_e \leq_T W_e^0$, W_e^2 is predicated on $l(e, s) \to \infty$ only, we must close our (e, i)-gap at the next e-expansionary stage $s_2 > s_1$. Of course, if s_2 does not occur we win $\hat{R}_e$. When s_2 occurs if none of the uses have changed we will need only re-impose restraint and pick a new follower. When s_2 occurs the desirable conditions are: that W_e^0 has permitted x, W_e^2 has permitted $\gamma^2(x, e, s)$ *and furthermore*

$$W_{e,s_1}^1[u(\Psi_{i,s_1}(W_{e,s_1}^1; x))] = W_{e,s_2}^2[u(\Psi_{i,s_1}(W_{e,s_1}^1; x))].$$

(After all we also need to know that at s_2, $L(e, i, s_2) > x$.) Should these conditions occur we enumerate x into Q_e, and raise $r(e, i, s_2) = s_2$ to preserve the disagreement $0 = \Psi_i(W_e^1; x) \neq Q_e(x) = 1$.

The reader should note that if W_e^0 permits x then one of W_e^2 or W_e^1 must permit on $\lambda^2(e, s)$ (as $\lambda^2(e, s) > u(\Phi_{e,s_1}^0(W_{e,s_1}^1 \oplus W_{e,s_1}^2; x)) = \Psi(\Phi_{e,s}^0(W_{e,s}^1 \oplus W_{e,s}^2; x))$, the last equality because A was restrained between stages s and s_1). Should we close our (e, i)-gap unsuccessfully, it will be convenient to reuse x as a follower of $R_{e,i}$, although we will also need another follower $y > x$ to attack with x simultaneously later.

The rough idea is this: If $R_{e,i}$ fails we'd like to argue that for all followers x of $R_{e,i}$ whenever W_e^0 permits on x it must be that—since good conditions don't occur—*either* W_e^2 doesn't permit $\lambda^2(e, s) = \lambda^2(e, s_1)$ (and so W_e^1 *does* permit $\lambda^2(e, s)$) *or* W_e^1 permits $u(\Psi_{i,t}(W_{e,t}^1; x))$. As we will see below, this idea fails under multiple attacks, but we shall modify γ^0, γ^1 and define a W_e^1-use function $\rho(x, e, i, s)$ (predicated on $l(e, i, s) \to \infty$) so that if $R_{e,i}$ fails then whenever W_e^0 permits on x, W_e^1 permits on $\rho(x, e, i, s)$. In this way to compute $W_e^0[x]$ we only need compute an e-expansionary stage t where $W_{e,t}^1[\rho(x, e, i, t)] = W_e^1[\rho(x, e, i, t)]$.

The above is a general outline of the overall shape of the construction. We now discuss the problems associated with actual implementation of this general strategy. It is very important that the reader understand these problems, and our solution to them, as these features are at the heart of the construction.

The problems all stem from the fact that our current approximations to use functions of $\Gamma^0(W_e^0) = \Gamma_e^2(W_e^2) = Q_e$ and $\Gamma(W_e^1) = W_e^0$ are simply not sensitive enough for *multiple* attacks on $R_{e,i}$ via x. Recall that our current approximations are $\gamma^0(x, 0, s) = x$, $\gamma^2(x, e, s) = \gamma^2(e, s) = \gamma^2(e, s_1)$ (by restraints on A) and $\rho(x, e, i, s) = \max\{\gamma^2(x, e, s), u(\Psi_{i,s_1}(W_{e,s_1}^1; x))\}$.

The inadequacy of ρ can be seen as follows. We need to argue that if we ever see W_e^0 and W_e^2 permit respectively γ^0 and γ^2 and W_e^1 not permit $\rho(x, e, i, t)$ we ought to win. Consider the situation as given in our outline. Suppose that at stage s_2 we close our (e, i)-gap unsuccessfully but we see W_e^0 permit $x = \gamma^0(x, e, s_1)$ and W_e^1 permit $\gamma^2(x, e, s_1)$ but W_e^2 remain fixed. Now the use $u(s_2)$ of the computation $\Phi_{e,s_2}^0(W_{e,s_2}^1 \oplus W_{e,s_2}^2; x)$ might be very much larger than it was at stage s_1. Indeed perhaps $u(s_2) > \max\{\gamma^2(x, e, s_1), u(\Psi_{i,s_2}(W_{e,s_2}; x))\}$ and, for example, $W_e^1[u(\Psi_{i,s_1}(W_{e,s_1}^1; x))]$ is unchanged. Note that *as* $W_{e,s_2}^2[\gamma^2(x, e, s_1)]$ *is unchanged, we cannot change* $\gamma^2(x, e, s_2)$ *from* $\gamma^2(x, e, s_1)$. The point of this is that at the closure of a *subsequent* (e, i)-gap — say at stage $s_3 > s_2$ — we might see $W_e^0[x]$ permit and $W_e^2[u(s_2)]$ permit, *but neither* $W_e^2[\gamma^2(x, e, s_2)]$ *nor* $W_e^1[\rho(x, e, i, s_2)]$ *permit.* Perhaps the relevant W_e^2-change occurs only on $\{z : \gamma^2(x, e, s_2) < z \leq u(s_2)\}$.

This obviously creates a very serious problem since we have a situation with a W_e^0-, W_e^2-change and no W_e^1-change — which should be a win — yet we can't win because of an earlier unsuccessful attack at x (so we can't 'use' the relevant W_e^2-change).

Our solution to this problem is to define $\rho(x, e, i, t)$ in such a way as to ensure that if we have a situation where $W_e^0[x]$ permits and so $W_e^2[\lambda^2(e, t)]$ permits then if

W_e^1 doesn't permit $\rho(x, e, i, t)$ we will win on some follower $y \geqslant x$ (although we may not win at x). To do this, we delay the definition of $\rho(x, e, i, t)$ until a stage t is found where $\rho(x, e, i, t)$ 'covers' a new follower. We do this as follows. In the situation above, the idea is that at stage s_2 although we close the (e, i)-gap *we don't define* $\rho(x, e, i, s_2)$ *at all but appoint a new follower* $y > s_2$ (setting $r(e, i, s_2) = s_2$).

We then wait until a stage $\hat{s}_1 > s_2$ where $L(e, i, \hat{s}) > y$ *and only then define*

$$\rho(x, e, i, \hat{s}_1) = \rho(y, e, i, \hat{s}_1) = \max\{\gamma^2(y, e, \hat{s}_1),\ u(\Psi_{i,\hat{s}_1}(W_{e,\hat{s}_1}; y))\}.$$

The reader should realise that this delay in the definition of ρ is fine from $R_{e,i}$'s point of view as we only need ρ if $\Psi_i(W_e^1) = Q_e$ so that $L(e, i, s) \to \infty$. By monotonicity this specifically ensures that also

$$\rho(x, e, i, \hat{s}_1) > \max\{u(s_2),\ u(\Psi_{i,\hat{s}}(W_{e,\hat{s}_1}; x))\}.$$

Note that this inequality follows as $\gamma^2(y, e, \hat{s}_1) = \lambda^2(e, \hat{s}_1) \geqslant l(e, \hat{s}_1) > u(\hat{s}_1) = u(s_2)$, the last equality by restraints. Now we open an $(e. i)$-gap at y (and x) as we did at stage s_1.

The crucial observation is that if we close our (e, i)-gap at stage $\hat{s}_2 > \hat{s}_1$ then if $W_e^0[x]$ changes, $W_e^2[u(s_2)]$ changes and $W_e^1[\rho(x, e, i, \hat{s}_1)]$ does not, then it also must be that $W_e^0[y]$ changes, $W_e^2[\gamma^2(y, e, \hat{s}_1)]$ changes and $W_e^1[\rho(y, e, i, \hat{s}_1)]$ does not. (The point is that y is in its 'initial attack' phase.) Hence we can cause a disagreement at y. Obviously if at $\hat{s}_2$ we have the same problem with y as we did with x we will delay both $\rho(x, e, i, t)$ and $\rho(y, e, i, t)$ for some $z > y$. It is essential to this process that such resetting occurs only finitely often. This is achieved by a cancellation process; essentially we cancell all followers z *targeted* for A with $x < z < \hat{s}_1$ when we set $\rho(x, e, i, \hat{s}_1)$ and furthermore when we enumerate $\hat{x}$ into A we always cancel all numbers q *targeted* for A with $q > \hat{x}$. (As usual these have lower priority.) The reader should note that this won't cancel followers targeted for Q.

(3.2) This cancellation process helps us in the following way. If we have an unsuccessful closure where *no permissions* occur, then there is no reason to, (and nor would we be able to) change γ^0, γ^2 and ρ. Thus the only reason we need to delay the definition of ρ as above is that some follower z with $z \leqslant s_1$ entered A. Now the cancellation process at stage $\hat{s}_1$ means that there are now no followers left alive below $\hat{s}_1$ except those $\leqslant s_1$ (that were already appointed at stage s_1 as we will see). Thus again we have ensured that the only numbers that would cause a change in γ^0, γ^2 or ρ for x (or y) are those numbers $\leqslant s_1$. This cancellation occurs each time we delay ρ, and hence we ensure ρ is delayed (and changed) at most s_1 times.

The way other P_j requirements live with this is that there will be infinitely many 'no permission' outcomes. In particular if at $\hat{s}_2$ there was no change we would pick a follower q for P_j with $q > \hat{s}_2$, and a follower z of $R_{e,i}$ with $y < \hat{s}_2 < q < z$. (Here we suppose $\langle e, i \rangle < j$.) Now when we see $L(e, i, t) > z$ we don't cancel q unless we are in a $\rho(y, e, i, t)$-delay situation. Diagram 2 might be helpful.

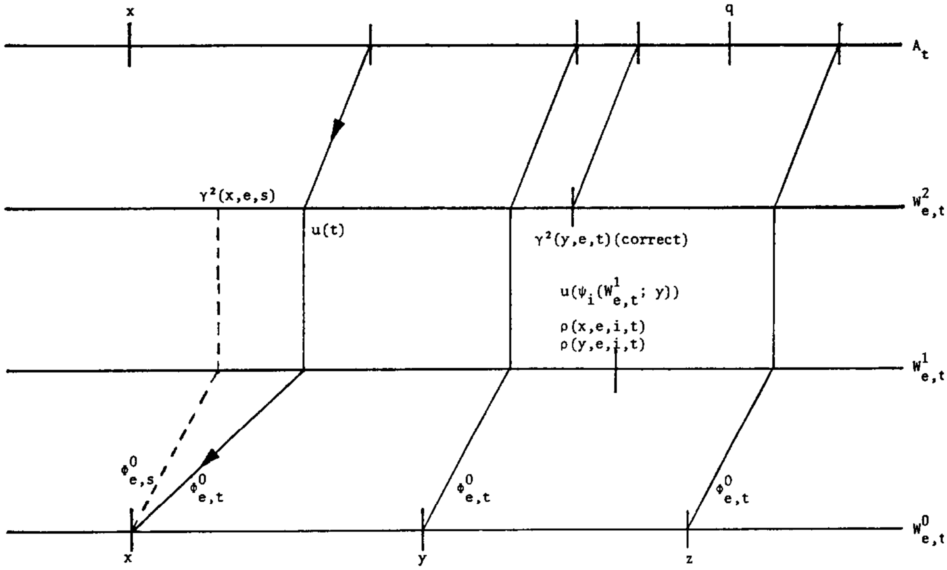

Diagram 2

Here we assume no change at s_2, q least and $u(\Psi_i) > \gamma^2$. Note that there are no followers alive between x and q. The lines between A_t and W_e^2 are meant to represent all the relevant reductions controlling W_e^i. Note we don't know where $u(\Psi_i(W_{e,t}; x))$ should lie, except that it must be inside $\rho(x, e, i, t)$.

Remember, the driving force in our construction is to ensure that whenever W_e^0 and W_e^2 permit γ^0 and γ^2 and W_e^1 doesn't permit ρ we win. The situation above deals with the possibility that γ^0 and ρ get to change, but γ^2 doesn't. The other situation that must be dealt with is not being able to move ρ although we don't win. With the current strategy, this can only occur if $W_e^2[u]$ permits—where $u = u(\Psi^0_{e,s_1}(W^1_{e,s_1} \oplus W^2_{e,s_1}; x))$—resetting $\gamma^2(x, e, s_1)$ but *both* $W_e^1[\rho(x, e, i, s_1)]$ and $W_e^0[x]$ $(= W_e^0[\gamma^0(x, e, s_1)]$ remain unchanged. (And so $W_e^1[u]$ unchanged too.) Under this scenario γ^2 might need to be very large when it is reset (necessarily) at stage s_2. Of course this creates no problem at stage s_2 as this is an unsuccessful closure. This creates a problem in the future since W_e^1 cannot comprehend (via ρ) this γ^2 change. That is, again at some stage $s_3 > s_2$ we might close another (e, i)-gap. But now we might see $W_e^0[x]$ change, but $W_e^0[\gamma^2(x, e, s_2)]$ not change (perhaps W_e^1 only changes on $\{z : \hat{u} \geq z > u\}$ where $\hat{u} = u(\Phi^0_{e,s_2}(W^1_{e,s_2} \oplus W^2_{e,s_2}; x))$. Now in this case we need some progress on '$W_e \leq_T W_e^1$' so it is desirable for W_e^1 to know (via $\rho(x, e, i, s_2)$) that $W_e^0[x]$ has changed. The trouble is that in this situation perhaps $W_e^1[\rho(x, e, i, s_2)]$ is unchanged.

Our solution to this last problem is to use a process we call *layering*. The first observation we need make is this: at stage s_2 *if there were no followers* $z < u(x, e, s_2)$ *left alive targeted for A this problem can't occur since the* $W_e^0[x]$ *computation is final.* (As $A_{s_2}[u(x, e, s_2)] = A(u(x, e, s_2))$].) Now, again if we see this situation (i.e. a bad unsuccessful closure) occurs, we will cancel all new followers (since s_1) so the comment above pertains if at stage s_1 there was only one follower $\leqslant x$ targeted for A. Suppose there are only two followers $z_1 < z_2 \leqslant x$ (and so only two $\leqslant u(x, e, s_1)$ by cancellation when we set γ^0 and γ^2). In this case we really are in trouble. Between stages s_1 and s_2, z_2 may enter A, killing γ^2. Then between stages s_2 and s_3 above z_1 may enter A putting us in the no win situation.

Our idea is to add one further *layer* to γ^0 and γ^1 to cope with z_1. That is, we don't define γ^0 and γ^2 until we see a stage s where not only is $l(e, s) > x$ but also $l(e, s) > u(x, e, s)$. Then as before we cancel all followers $> x$ targeted for A, to get the situation in Diagram 3.

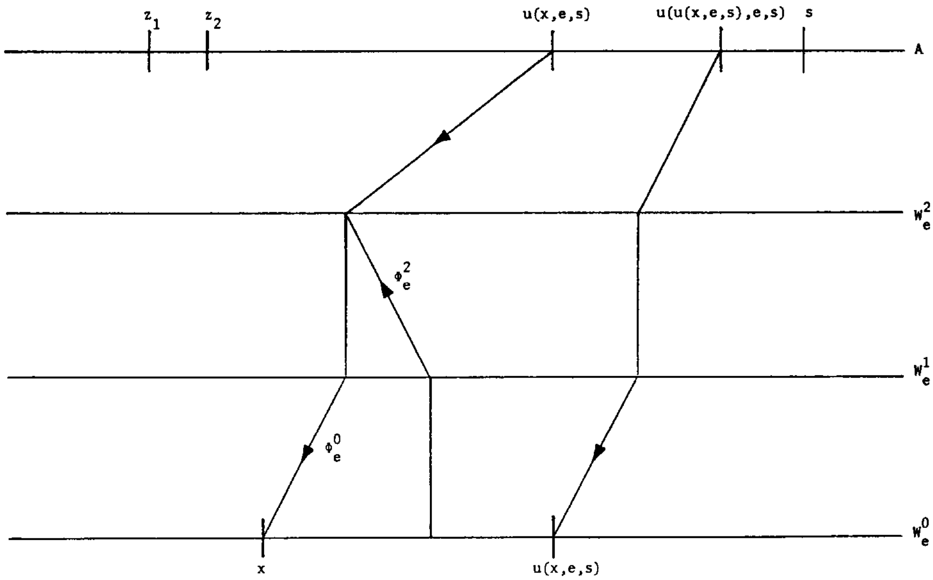

Diagram 3

We freeze this by setting (again) $r(e, i, s) = s$ but now the twist is to set $\gamma^0 = \lambda^0(x, s)$ and $\gamma^2 = \lambda^2(e, s)$. And at the stage s_1 where $L(e, i, s_1) > x$ we set $\rho(x, e, i, s_1) = \max\{u(\Psi_{i,s_1}(W^1_{e,s_1}; x)),\ \gamma^0(x, e, s_1),\ \gamma^2(x, e, s_1)\}$. The point of this procedure is this. Suppose at stage s_2 we get our bad outcome: that $W_e^0[\gamma^0]$ and $W_e^1[\rho]$ both *don't* change but $W_e^2[\gamma^2]$ does. The only way this can occur is for $W_e^2[\gamma^2]$ to change only on the 'outer layer'. That is W_e^2 can only change on

$$\{y : \gamma^2(x, e, s) \geqslant y > u = u(\Phi_e^0(W^1_{e,s} \oplus W^2_{e,s}; x))\},$$

since if W_e^2 changes below u then $W_e^0 \oplus W_e^1$ must change below $\lambda^0(e, s)$. As $W_e^1[\rho]$

is unchanged this can only mean that $W_e^0[\lambda^0(e, s)]$ changes, contradiction. Hence we get to reset γ^2 but not ρ, but know that the 'inner layer' $W_e^2 \oplus W_e^1[u]$ is unchanged. At stage s_2 we cancel all followers y targeted for A with $z_1 < y < s_2$. Therefore, we know that there is only one more possible change to $W_e^0[x]$ (caused by z_1). The whole point is that if at some gap $s_3 > s_2$, $W_{e,s_3}^0[x]$ changes, it must cause a change in $W_e^1[u]$ or $W_e^2[u]$ and hence in either $W_e^1[\rho(e, s, i, s_2)]$ or $W_e^2[\gamma^2(x, e, s_2)]$. Therefore, by the same argument as we had—for only one possible injury instead of two—we can see $W_e^0 \leqslant_T W_e^1$ if the attack is unsuccessful.

In general the idea is to make sure that there are more layers in the uses $\gamma^0(x, e, s)$ and $\gamma^2(x, e, s)$ than there are possible injuries to these layers. We can do this by our cancellation procedures at bad closures and since we know that x layers will suffice. (The point is that this sort of injury only strips one layer per gap.)

In summary, at the close of an (e, i)-gap at stage s_1 if it is unsuccessful it is not the case that $W_e^0[\gamma^0]$ and $W_e^2[\gamma^2]$ change but $W_e^1[\rho]$ doesn't. If W_e^0 and W_e^2 change so that W_e^1 also changes, we can reset all of γ^0, γ^2 and ρ. If W_e^j changes and W_e^{j+2} doesn't change for $j \in \{0, 2\}$ and also W_e^1 changes, we can reset ρ to some $\rho(y, e, i, t)$ for some $y > x$. This comment also applies only if ρ changes, that is if only W_e^1 changes.

Finally if only one W_e^i for $i \in \{0, 2\}$ changes and W_e^1 doesn't, changes must occur only on the outermost live layer. We only get to reset γ^i but are safe in the knowledge that we have more layers than injuries. This is all formalised in the basic module below.

The basic module. Implement the following steps.

Step 1. Pick an initial follower $x_0 = x(e, 0, s_0) = \langle e, s_0 \rangle$ or inductively some follower $x_j = \langle e, s_0 \rangle$ at stage s_0. (In the α-module 'e' will be replaced by a string σ from the priority tree.)

Step 2. At the first stage $s > s_0$ where we see $l^{x_j}(e, s) > x_j$ cancel all followers y targeted for A with $y > x_j$. Set $r(e, i, s) = s$. Here we define $l^{x_j}(e, s)$ inductively via

$$(3.3)\qquad \begin{cases} l^{x_j}(e, s) = \max\{z : l(e, s) > u^{x_j}(z, e, s)\}, & \text{where} \\ u^0(z, e, s) = u(z, e, s), & \text{and for} \\ l(e, s) > u^i(z, e, s) & \text{we define} \\ u^{i+1}(z, e, s) = u(u^i(z, e, s), e, s)). \end{cases}$$

This creates a barrier with $> x_j$ layers. Now define $\gamma^k(x_j, e, s) = \lambda^k(x_j, e, s)$ for $k = 0, 2$. Put the basic module in state w (for 'wait').

Step 3. If x_j is the largest follower of $R_{e,i}$ and $L(e, i, s_1) > x_j$ for some least $s_1 > s$ then to each x_k for $k \leqslant j$ with $\rho(x_k, e, i, s_1 - 1)$ undefined assign $\rho(x_k, e, i, s_1) = \max\{\lambda^2(e, s_1), \lambda^1(e, s_1)\}$. If $\rho(x_{j-1}, e, i, s_1 - 1)$ is undefined, cancel all followers y targeted for A with $y > x_{j-1}$. Now open an (e, i)-gap by setting $r(e, i, s_1) = 0$, and (so) put the basic module into stage g.

Step 4. At the least stage $s_2 > s_1$ where $l(e, s_2) > ml(e, s_2)$ and $l^{x_j}(e, s_2) > x_j$ close the (e, i)-gap by setting $r(e, i, s_2) = s_2$. Adopt the first case below to pertain

Case 1 (*successful closure*). For some $k \leqslant j$, $W^0_{e,s_2}[\gamma^0(x_k, e, s_1)] \neq W^0_{e,s_1}[\gamma^0(x_k, e, s_1)]$ and $W^2_{e,s_2}[\gamma^2(x_k, e, s_1)] \neq W^2_{e,s_1}[\gamma^2(x_k, e, s_1)]$ but $W^1_{e,s_2}[\rho(x_k, e, i, s_1)] = W^1_{e,s_1}[\rho(x_k, e, i, s_1)]$.

Action. Set $Q_{e,s_2+1} = Q_{e,s_2} \cup \{x_k\}$. Put the basic module into state f (for 'finish').

Case 2 (*unsuccessful closure*). Otherwise. This is outcome g and is where we possibly have appointed new followers to P_j for $j > \langle e, i\rangle$ before we appoint followers to $R_{\langle e,i\rangle}$. (More on the coherence later.) For each $k \leqslant j$ adopt the first case to pertain.

For convenience, set $\gamma^q(x_k) = \gamma^q(x_k, e, s_1)$ for $q \in \{0, 2\}$ and $\rho(x_k) = \rho(x_k, e, i, s_1)$.

Case 2(a). $W^q_{e,s_2}[\gamma^q(x_k)] = W^q_{e,s_1}[\gamma^q(x_k)]$ for $q \in \{0, 2\}$ and $W^1_{e,s_2}[\rho(x_k)] = W^1_{e,s_1}[\rho(x_k)]$.

Action. No change.

Case 2(b). $W^q_{e,s_2}[\gamma^q(x_k)] = W^q_{e,s_1}[\gamma^q(x_k)]$ for at least one of $q = 0$ or $q = 2$ but $W^1_{e,s_2}[\rho(x_k)] \neq W^1_{e,s_1}[\rho(x_k)]$.

Action. If $W^q_{e,s_2}[\gamma^q(x_k)]$ has changed, then set $\gamma^q(x_k, e, s_2) = \lambda^q(e, s_2)$. Declare $\rho(x_k, e, i, s_2)$ as undefined (pending for the new follower x_{j+1}) unless there is some follower $x_{\hat{k}}$ for $\hat{k} > k$ to which case 2(c) below pertains. In that case we set $\rho(x_k, e, i, s_2) = \rho(x_{\hat{k}}, e, i, s_2)$ for the least such $\hat{k}$. In any case cancel all followers y targeted for A with $y > x_k$.

Case 2(c). $W^q_{e,s_2}[\gamma^q(x_k)] \neq W^q_{e,s_1}[\gamma^q(x_k)]$ for both $q = 0$ and $q = 2$ and $W^1_{e,s_2}[\rho(x_k)] \neq W^1_{e,s_1}[(x_k)]$ but $L(e, i, s_2) > x_k$.

Action. Set $\gamma^q(x_k, e, s_2) = \lambda^q(e, s_2)$ and $\rho(x_k, e, i, s_2) = \max\{\lambda^2(e, s_1), \lambda^1(e, s_1)\}$. Cancel all followers y targeted for A with $y > x_k$.

Case 2(d). As in case 2(c) but $L(e, i, s_2) \not> x_k$.

Action. Set $\gamma^q(x_k, e, s_2) = \lambda^q(e, s_2)$ but $\rho(x_k, e, i, s_2)$ is declared undefined (pending x_{j+1}). Cancel as in case 2(c).

Case 2(e). For some $q = 0$ or $q = 2$, $W^q_{e,s_2}[\gamma^q(x_k)] \neq W^q_{e,s_1}[\gamma^q(x_k)]$ but $W^{q+2}_{e,s_2}[\gamma^{q+2}(x_k)] = W^{q+2}_{e,s_1}[\gamma^{q+2}(x_k)]$ and case 2(b) did not pertain so that $W^1_{e,s_2}[\rho(x_k)] = W^1_{e,s_1}[\rho(x_k)]$.

Action. Set $\gamma^q(x_k, e, s_2) = \lambda^q(e, s_2)$ for this q. Cancel as in case 2(c). This case involves 'layer injury'.

(3.4) **Remark.** The reader should note that in the construction to follow, P_j (for $j > \langle e, i\rangle$) cooperates with R_e by assigning followers during the gap. However, these new followers will be cancelled should any case except 2(a) pertain. The point is that if g is the correct outcome, case 2(a) pertains infinitely often due to our cancellation process we described earlier (as we shall see). If case 2(a) does not pertain, the $R_{e,i}$ module only has finite effect. (More on this later.) The remaining details in the full construction contain no surprises and fit into the

$\mathbf{0}'''$-framework of Slaman/Soare in Soare [13]. We expect readers familiar with this (or similar) accounts may wish to supply them for themselves. For completeness, we give some formal details below.

(3.5) **The priority tree.** Let $\Lambda = \{f, g, w\}$ and order Λ by $f <_\Lambda g <_\Lambda w$. Define the priority tree T as the collection of all strings σ with $\sigma \in \{f, g, w, 0, 1\}^*$ such that if $j \equiv 1 \pmod 3$ or $j \equiv 0 \pmod 3$ then $\sigma(j) \in \{0, 1\}$. Otherwise $\sigma(j) \in \{f, g, w\}$. For $\sigma, \tau \in T$ let $\sigma \subseteq \tau$ denote σ being an initial segment of τ. Let $\leq_{\mathrm{L}}$ be lexicographic ordering. Thus $\sigma \leq_{\mathrm{L}} \tau$ means either $\sigma \subseteq \tau$ or $(\exists \gamma)[(\gamma\hat{\ }0 \subseteq \sigma \,\&\, \gamma\hat{\ }1 \subseteq \tau) \vee [\gamma\hat{\ }f \subseteq \sigma \,\&\, (\gamma\hat{\ }g \subseteq \tau \vee \gamma\hat{\ }w \subseteq \tau)] \vee (\gamma\hat{\ }g \subseteq \sigma \,\&\, \gamma\hat{\ }w \subseteq \tau)]$. We refer to $\sigma, \tau \in T$ as *guesses*.

For a guess σ, let $\mathrm{lh}(\sigma)$ denote the length of σ. If $\mathrm{lh}(\sigma) \equiv 0 \pmod 3$, then σ is devoted to solving $\hat{R}_e$ where $\mathrm{lh}(\sigma) = 3e$. If $\mathrm{lh}(\sigma) = 3e + 1$, then σ is devoted to solving P_e. Finally if $\mathrm{lh}(\sigma) \equiv 2 \pmod 3$, then σ is devoted to some $R_{e,i}$ as determined by the list below.

(3.6) **The list and priority assignments.** We assign priorities by induction on $\mathrm{lh}(\sigma)$ then on $\leq_{\mathrm{L}}$. We shall have two partial functions e and i which map $T \to \omega$ and a list L below. We regard $\langle , \rangle$ as having $e < \langle e, j\rangle$ for all j. Let $n = \mathrm{lh}(\alpha)$.

$n = 0$. Define, as above, $e(\lambda) = 0$ and set $L(\lambda) = \omega$.

$n > 1$. Let $\alpha = \sigma\hat{\ }k$ for $k \in \Lambda \cup \{1, 0\}$ and assume $L(\sigma)$ defined.

Case 1: $\mathrm{lh}(\sigma) = 3e$. Define $e(\alpha) = e$ and adopt the first subcase below.

Subcase (i): $k = 0$. $L(\alpha) = L(\sigma)$.

Subcase (ii): $k = 1$. $L(\alpha) = L(\sigma) - \{\langle e(\sigma), j\rangle : j \in \omega\}$.

Case 2: $\mathrm{lh}(\sigma) = 3e + 1$. Let $L(\alpha) = L(\sigma)$, and define $\langle e(\alpha), i(\alpha)\rangle = \mu z\ (z \in L(\alpha))$.

Case 3: $\mathrm{lh}(\sigma) = 3e + 2$. Define $e(\alpha) = e + 1$. Adopt the first case below to pertain.

Subcase (i): $k = f$ or $k = w$. Set $L(\alpha) = L(\sigma) - \{\langle e(\sigma), i(\sigma)\rangle\}$.

Subcase (ii): $k = g$. Set $L(\alpha) = L(\sigma) - \{\langle e(\sigma), j\rangle : j \in \omega\}$.

This concludes the priority assignment.

(3.7) **The regions.** Fix $\alpha \in T$ with $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$ so that $\langle e(\alpha), i(\alpha)\rangle$ is defined, and α is devoted to solving $R_{e(\alpha), i(\alpha)}$. Define the *top of the* $e(\alpha)$*-region containing* α, $\tau(\alpha)$ via

$$\tau(\alpha) = (\mu\sigma \subset \alpha)(\mathrm{lh}(\sigma) \equiv 0 \pmod 3 \,\&\, e(\alpha) = e(\sigma)).$$

With this, define the *e-region containing* α as

$$E(\alpha, e) = \{\sigma : \sigma \in T \,\&\, \tau(\alpha) \subset \sigma\}.$$

For guesses $\sigma \in T$ with $\mathrm{lh}(\sigma) \equiv 2 \pmod 3$ there will be followers denoted by $x(\sigma, j, s)$ for $j \in \omega$. If the outcome is g (so that $\sigma\hat{\ }g$ is on the true path), then $\lim_s x(\alpha, j, s) = x(\alpha, j)$ exists and will be recursive. P_e will have followers with various guesses. If $e(\alpha) = e$, then a follower of P_e at guess α (i.e. a follower of P_α) is denoted by $y(\alpha, s)$. If $\mathrm{lh}(\sigma) \equiv 2 \pmod 3$, we will define a restraint $r(\alpha, s)$, and a use $\rho(x, \alpha, s)$ $(= \rho(x, e, i, s)$ at $\alpha)$ for $x = x(\alpha, j, s)$ and also for either $\eta = \alpha$ or $\eta = \tau(\alpha)$ we simultaneously define $\gamma^p(x, \eta, s)$ for $p = 0, 2$. These are the guessed versions of the uses of the basic module. To *initialize* node α at stage s we mean, as usual, to cancel all followers, restraints etc. associated with α, reset the current state of the α-module at node α (denoted by $F(\alpha, s)$) to $F(\alpha, s) = w$ if $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$, cancel Q_α (if defined) to $Q_\alpha = \emptyset$ and cancel any links (to be defined) with top or bottom α.

(3.8) **Definition.** Let $\alpha \in T$.

(i) We say $s + 1$ is an *α-stage* if $\alpha \subset \sigma_{s+1}$ where σ_{s+1} is to be defined later. In addition 0 is an α-stage.

(ii) We say $s + 1$ is a *genuine α-stage* if $\sigma(t, s + 1) = \alpha$ for some substage t of stage $s + 1$. We let G^α denote the collection of genuine α-stages.

(iii) Suppose $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ with $\mathrm{lh}(\alpha) = 3e$ so $e = e(\alpha)$. We say that a stage q is *α-expansionary* if $q = 0$ or $q = s + 1$ where

(a) $s + 1$ is a genuine α-stage,

(b) $l(e, q) > \max\{l(e, \hat{q}) : \hat{q}$ is an α-expansionary stage and $\hat{q} < q\}$, and

(c) For all followers x of the form $x = x(\gamma, j, \hat{q} - 1)$ with $\gamma \supset \alpha$, $\hat{q} \leq q$ and $\tau(\gamma) = \alpha$, $l^x(e, q) > x$ (where $l^x(e, q)$ is defined as in (3.3)).

(iv) Suppose that $\mathrm{lh}(\alpha) = 3e + 1$. We say that α *requires attention at substage t of stage $s + 1$* (which we write as *stage* $(t, s + 1)$) if $W_{e,s} \cap A_s = \emptyset$ and one of the following options holds:

(a) $\sigma(t, s + 1) = \alpha$ and $y(\alpha, s)$ is undefined,

(b) $\sigma(t, s + 1) = \alpha$ and $y(\alpha, s) \in W_{e,s}$.

(v) Suppose that $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$. Let $e = e(\alpha)$ and $i = i(\alpha)$. We say that a stage q is *α-expansionary* if $q = 0$ or $q = s + 1$ and

(a) $s + 1$ is a genuine α-stage,

(b) $l(e, i, q) > \max\{l(e, i, \hat{q}) : \hat{q}$ is an α-expansionary stage $< q\}$.

(c) For all followers $x(\alpha, i, s)$ currently defined, $L(e, i, q) > x(\alpha, i, s)$.

(vi) Suppose that $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$. We say that α *requires attention at stage* $s + 1$ if $s + 1$ is a genuine α-stage and α has no follower (i.e. $x(\alpha, 0, s)$ is undefined).

(3.9) **The Construction**

Stage 0. Initialize all $\alpha \in T$. Define $\sigma_0 = \lambda$.

Stage $s + 1$. The value of a parameter $p \neq \sigma$ at stage $(t, s + 1)$ is denoted by p_t.

Substage $t = 0$. Define $\sigma(0, s + 1) = \lambda$.

Substage $t + 1$. We are given $\sigma(t, s + 1)$ and for all $\alpha \in T$ with $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$, $F_t(\alpha) = F_t(\alpha, s + 1)$, with $F_t(\alpha) \in \{w, g, f\}$. Adopt the first case below to pertain. Let $\alpha = \sigma(t, s + 1)$.

Case 1: $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$.

Subcase 1: Stage $s+1$ is not α-expansionary. Define $\sigma(t+1, s+1) = \alpha\hat{\ }1$. Go to stage $(t+2, s+1)$ unless $t = s$. If $t = s$, set $\sigma_{s+1} = \sigma(t+1, s+1)$ and initialize all $\tau \not\leqslant_{\mathrm{L}} \sigma_{s+1}$ and go to stage $s+2$.

Subcase 2: Stage $s+1$ is α-expansionary and there is no link (α, ρ) defined at stage $(t, s+1)$. In this case define $\sigma(t+1, s+1) = \alpha\hat{\ }0$. Go to stage $(t+1, s+1)$ unless $t = s$. If $t = s$, set $\sigma_{s+1} = \sigma(t+1, s+1)$ and initialize all $\tau \not\leqslant_{\mathrm{L}} \sigma_{s+1}$ and go to stage $s+2$.

Subcase 3: Stage $s+1$ is α-expansionary and there is a link (α, ρ) defined at stage $(t, s+1)$. In this case define $\sigma(t+1, s+1) = \rho$. (It won't be that $t = s$.) Go to stage $(t+2, s+1)$.

Case 2: $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$. Adopt the first case below to pertain.

Subcase 1: α does not require attention. Set $\sigma(t+1, s+1) = \alpha\hat{\ }0$ and go to stage $(t+2, s+1)$ unless $t = s$, in which case set $\sigma_{s+1} = \alpha\hat{\ }0$, initialize all $\tau \not\leqslant_{\mathrm{L}} \sigma_{s+1}$ and go to stage $s+2$.

Subcase 2: α requires attention. If $y(\alpha, s)$ is not defined, set $y(\alpha, s+1) = \langle \alpha, s+1 \rangle$ and $\sigma(t+1, s+1) = \alpha\hat{\ }1$. If $y(\alpha, s+1)$ defined (so that (3.8)(iv)(b) holds) enumerate $y(\alpha, s)$ into $A_{s+1} - A_s$, and define $\sigma(t+1, s+1) = \alpha\hat{\ }0$. In either case set $\sigma_{s+1} = \sigma(t+1, s+1)$, initialize all $\tau \not\leqslant_{\mathrm{L}} \sigma_{s+1}$ and go to stage $s+2$.

Case 3: $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$. Let $e = e(\alpha)$ and $i = i(\alpha)$. Adopt the first subcase to pertain.

Subcase 1: $F_t(\alpha, s+1) = f$. Define $\sigma(t+1, s+1) = \alpha\hat{\ }f$. (It is not possible that $t = s$ here.) Go to stage $(t+2, s+1)$.

Subcase 2: $F_t(\alpha, s+1) = w$ and α requires attention. Define $x(\alpha, 0, s+1) = \langle \alpha, s+1 \rangle$ and $\sigma_{s+1} = \sigma(t+1, s+1) = \alpha\hat{\ }w$. Initialize all $\gamma \not\leqslant_{\mathrm{L}} \sigma_{s+1}$. Create a link $(\tau(\alpha), \alpha)$. Keep $F(\alpha, s+1) = w$. Go to stage $s+2$.

Subcase 3: $F_t(\alpha, s+1) = w$ and we have just travelled a link (τ, α) (and hence $\sigma(t-1, s+1) = \tau = \tau(\alpha)$). Let $x = x(\alpha, j, s)$ be the largest defined follower of α. Define $\gamma^k(x, \alpha, s+1) = \lambda^k(e, s+1)$ for $k = 0, 2$, $r(\alpha, s+1) = s+1$ and cancel all followers y targeted for A with $y > x$. Let $\sigma_{s+1} = \sigma(t+1, s+1) = \alpha\hat{\ }w$. Now for all nodes γ with $\sigma_{s+1} \leqslant_{\mathrm{L}} \gamma$ but $\sigma_{s+1} \not\subseteq \gamma$, initialize γ. (Note that this is a different cancellation procedure than in the other cases.) Remove the link (τ, α) and go to stage $s+2$.

Subcase 4: $F_t(\alpha, s+1) = w$, α does not require attention, subcase 3 did not pertain and $s+1$ is not α-expansionary. Define $\sigma(t+1, s+1) = \alpha\hat{\ }w$, $F_{t+1}(\alpha, s+1) = w$ and go to stage $(t+2, s+1)$.

Subcase 5: $F_t(\alpha, s+1) = w$ and none of the above pertain (so that $s+1$ is α-expansionary). Define $\sigma(t+1, s+1) = \alpha\hat{\ }g$ and *open an α-gap*. Create a link $(\tau(\alpha), \alpha)$. Set $r(\alpha, s+1) = 0$. Let $x = x(\alpha, j, s)$ be the largest defined follower of α. For each $\hat{\jmath} \leqslant j$ with $\rho(x(\alpha, \hat{\jmath}, s), \alpha, s)$ undefined define

$$\rho(x(\alpha, \hat{\jmath}, s), \alpha, s+1) = \rho(x, \alpha, s+1) = \max\{\lambda^2(e, s+1), \lambda^1(e, s+1)\}.$$

(Compare with step 3 (of the basic module).) Go to substep $t+2$, setting $F_{t+1}(\alpha, s+1)=g$.

Subcase 6: $F_t(\alpha, s+1)=g$. (This will mean that we have just travelled a link (τ, α).) Define $r(\alpha, s+1)=\langle\alpha, s+1\rangle$ and close the α-gap. Remove the link (τ, α) unless case 2 pertains. Let $\hat{s}$ be the stage where the α-gap was opened. Adopt the first subcase below which pertains for each currently defined follower $x_k=x(\alpha, k, s)$.

Case 1 (*successful closure*). $W^p_{e,s+1}[\gamma^p(x_k, \alpha, \hat{s})]\neq W^p_{e,\hat{s}}[\gamma^p(x_k, \alpha, \hat{s})]$ for $p=0$, 2 but $W^1_{e,s+1}[\rho(x_k, \alpha, \hat{s})]=W^1_{e,\hat{s}}[\rho(x_k, \alpha, \hat{s})]$.

Action. Set $Q_{\tau(\alpha),s+1}=Q_{\tau(\alpha),s}\cup\{x_k\}$. Cancel all followers $x(\alpha, \hat{k}, s)$ for $\hat{k}\neq k$, define $\sigma(t+1, s+1)=\alpha\hat{}f=\sigma_{s+1}$ and $F(\alpha, s+1)=f$. Initialize all $\gamma\nleq_{\mathrm{L}}\sigma_{s+1}$. Go to stage $s+2$.

Case 2 (*unsuccessful closure*). Otherwise. For $q\in\{0,2\}$ if $W^q_{e,s+1}[\gamma^q(x_k)]\neq W^q_{e,\hat{s}}[\gamma^q(x_k)]$ (where $\gamma^q(x_k)=\gamma^q(x_k, \alpha, \hat{s})$ and similarly $\rho(x_k)$) define $\gamma^q(x_k, \alpha, s+1)=\lambda^q(e, s+1)$. For any such k cancel all followers y targeted for A with $y>x_k$. Now, in any case, find a large fresh number z exceeding all previously seen numbers. Without loss (by delay if necessary) we may suppose $z=\langle\alpha, s+1\rangle$. Let j be largest with $x(\alpha, j, s)$ defined. Define $x(\alpha, j+1, s+1)=\langle\alpha, s+1\rangle$. Note that we keep the link (τ, α) where $\tau=\tau(\alpha)$. Define $\sigma_{s+1}=\alpha\hat{}w$ and initialize all γ with $\sigma_{s+1}\leq_{\mathrm{L}}\gamma$ and $\sigma_{s+1}\not\subset\gamma$. (Again this is not the same initialization as for a successful closure.) Now adopt the first case below to pertain (for each k), and then go to stage $s+2$.

Case 2(a). For $q\in\{0,2\}$, $W^q_{e,s+1}[\gamma^q(x_k)]=W^q_{e,\hat{s}}[\gamma^q(x_k)]$ and $W^1_{e,s+1}[\rho(x_k)]=W^1_{e,\hat{s}}[\rho(x_k)]$.

Action. Do nothing (else).

Case 2(b). $W^q_{e,s+1}[\gamma^q(x_k)]=W^q_{e,\hat{s}}[\gamma^q(x_k)]$ for at least one $q\in\{0,2\}$ but $W^1_{e,s+1}[\rho(x_k)]\neq W^1_{e,\hat{s}}[\rho(x_k)]$.

Action. Declare $\rho(x_k, \alpha, s+1)$ as undefined pending x_{j+1} unless there is some follower $x_{\hat{k}}$ for $\hat{k}>k$ where case 2(c) below pertains. In this latter case set $\rho(x_k, \alpha, s+1)=\rho(x_{\hat{k}}, \alpha, s+1)$ for the least such $\hat{k}$.

Case 2(c). $W^q_{e,s+1}[\gamma^q(x_k)]\neq W^q_{e,\hat{s}}[\gamma^q(x_k)]$ for $q\in\{0,2\}$. $W^1_{e,s+1}[\rho(x_k)]\neq W^1_{e,\hat{s}}[\rho(x_k)]$ and $L(e, i, s+1)>x_k$.

Action. Set $\rho(x_k, e, i, s+1)=\max\{\lambda^2(e, s+1), \lambda^1(e, s+1)\}$.

Case 2(d). As in case 2(c) but $L(e, i, s+1)\not> x_k$.

Action. Declare $\rho(x_k, e, i, s+1)$ as undefined (pending x_{j+1}).

Case 2(e). For some $q=0$ or $q=2$, $W^q_{e,s+1}[\gamma^q(x_k)]\neq W^q_{e,\hat{s}}[\gamma^q(x_k)]$ but $W^{q+2}_{e,s+1}[\gamma^{q+2}(x_k)]=W^{q+2}_{e,\hat{s}}[\gamma^{q+2}(x_k)]$ and case 2(b) did not pertain (so $W^1_{e,s+1}[\rho(x_k)]=W^1_{e,\hat{s}}[\rho(x_k)]$.

Action. Do nothing (else).

Remark. The reader should note that cases 2(a) and 2(e) differ in that in case 2(a) followers $y>x_k$ targeted for A are not cancelled (as no γ^q is reset); yet in case 2(e) $\gamma^q(x_k, \alpha, s+1)$ is reset to $\lambda^q(e, s+1)$ and so followers are cancelled. □ End of Construction

(3.10) **Verification.** We shall use the following technical lemma whose proof is an easy induction on the construction, which we leave to the reader. We use it implicitly.

(3.11) **Lemma.** (i) *If* (τ, α) *is a link, then* $\alpha \supset \tau\hat{\ }0$, $e(\alpha) = e(\tau)$, $\tau = \tau(\alpha)$, $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$ *and* $\mathrm{lh}(\tau) \equiv 0 \pmod 3$.

(ii) *For all* $\alpha \supset \tau$ *if* $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$, *then* $e(\alpha) > e(\tau)$.

(iii) *There is at most one link* (τ, α) *with bottom* α *or top* τ *existing at the end of any substage.*

(iv) *Any link* (τ, α) *may be travelled at most twice before it is removed. Furthermore, if* (τ, α) *is a link travelled at stage s and* (τ, α) *exists at the end of stage s, then s is an unsuccessful closure of an* α*-gap. In this case, the link* (τ, α) *will be removed at the next* τ*-expansionary stage* $\hat{s}$ *(if* $\hat{s}$ *exists).*

(v) *Suppose that* (τ_1, α_1) *and* (τ_2, α_2) *are links both existing at the end of stage s. Suppose that* (τ_2, α_2) *is created at stage s, but* (τ_1, α_1) *already exists at stage s. Then s is not* τ_1*-expansionary,* $\tau_1\hat{\ }0 \not\subseteq \tau_2$ *but either* $\tau_1 \leqslant_L \tau_2$ *or* $\tau_2 \subsetneqq \tau_1$. *In either case, if* (τ_1, α_1) *is travelled at some stage* $t > s$, *then either* (τ_2, α_2) *is cancelled at stage t or* (τ_2, α_2) *has been removed at stage t. Finally, if* τ_2 *is not initialized at stage t, then* $\tau_2 \subset \tau_1$ *and hence* $e(\tau_2) < e(\tau_1)$.

(vi) *Suppose that* (τ_1, α_1) *and* (τ_2, α_2) *are links existing at the end of stage s with* $\tau_1 \subsetneqq \tau_2$ *(so that* $e(\tau_1) < e(\tau_2)$*). Then, if* (τ_2, α_2) *is created at stage* s_2 *and* (τ_1, α_1) *at stage* s_1, *we have* $s_1 \leqslant s_2$. *Furthermore* (τ_1, α_1) *must be removed before* (τ_2, α_2).

Let β denote the leftmost path. That is, the leftmost $\beta \in [T]$ such that for all strings $\sigma \in T$ if $\sigma \leqslant_L \beta$ and $\sigma \not\subseteq \beta$ then there are only finitely many stages where $\sigma \subseteq \sigma_s$. The following (fairly routine) lemma establishes that β is 'genuine'.

(3.12) **Lemma.** *Fix n and* $\alpha \subset \beta$ *with* $\mathrm{lh}(\alpha) = n$. *Then*

(i) $\exists^\infty s\,(\alpha \subsetneqq \sigma_s)$.

(ii) $|G^\alpha| = \infty$.

(iii) *If* $\mathrm{lh}(\alpha) = 1 \pmod 3$, *then* α *receives attention only finitely often.*

(iv) $(\forall \gamma \subset \alpha)$ *If* $\mathrm{lh}(\gamma) \equiv 0 \pmod 3$ *and* $\gamma\hat{\ }0 \subset \beta$, *then every link with top* γ *is removed and there are infinitely many genuine* γ*-stages where there are no links with top* γ.

Proof. All except (iv) clearly hold for $n = 0$. To see (iv) if $0 \subset \beta$ then there are infinitely many 0-expansionary stages. Suppose (λ, ρ) is any link at stage t, say. At the next 0-expansionary stage s if (λ, ρ) is not yet cancelled, it must be travelled and thus $\sigma(1, s) = \rho$. Let $\hat{s}$ be the stage where the link was created. Then at $\hat{s}$ case 3 pertained to ρ via one of subcase 2, subcase 5 or subcase 6 held for ρ. If subcase 2 or 6 pertained, then the link is removed at stage s.

If subcase 5 pertained then subcase 6 now pertains to ρ. Thus by the next 0-expansionary stage exceeding s we will have removed (λ, ρ) if it is not removed at stage s. Note that the next λ-stage will (be genuine) and there will be no links with top λ.

Now suppose the lemma for $n \geq 0$ and let $\gamma \subset \beta$ with $\mathrm{lh}(\gamma) = n$. Let $\alpha = \gamma\hat{\ }q$ for $\alpha \subset \beta$. Let s_1 be a stage such that for all $s > s_1$:

(a) $\sigma_s \leq_{\mathrm{L}} \alpha$ implies $\sigma_s \subset \alpha$.

(b) If $\mathrm{lh}(\hat{\gamma}) \equiv 1 \pmod 3$ and $\hat{\gamma} \subseteq \gamma$, then $\hat{\gamma}$ does not receive attention at stage s.

(c) If $\mathrm{lh}(\hat{\gamma}) \equiv 2 \pmod 3$ and $\hat{\gamma}\hat{\ }f \subseteq \gamma$, then $\hat{\gamma}$ does not receive attention at stage s.

(d) If $q = 1$ and $\mathrm{lh}(\gamma) = 0 \pmod 3$, then s is not γ-expansionary.

(e) If $q = w$ and $\mathrm{lh}(\gamma) = 2 \pmod 3$, then s is not γ-expansionary.

(f) If $q = f$ and $\mathrm{lh}(\gamma) = 2 \pmod 3$, then γ does not require attention at stage s.

Now suppose the lemma for all $\hat{\gamma} \subseteq \gamma$. To see (ii) and (iv) let $s_2 > s_1$ be any stage. By hypothesis, there is a genuine γ-stage $s_3 > s_2$ say $\gamma = \sigma(t, s_3)$. Now if s_3 is not a genuine α-stage there are only two possibilities: either $\sigma_{s_3} = \gamma$ or there is a link (γ, ρ). In the second case if (γ, ρ) is not removed or cancelled at stage s_3, then it will be at stage $\hat{s}_3 > s_3$, the least genuine γ-stage after s_3, by the same reasoning as we used for λ. In any case it follows that at the first genuine $\alpha = \gamma\hat{\ }0$-stage exceeding s_3 there must be no link with top γ and this will be a genuine α stage. Thus we get (ii) and (iv).

Finally to get (i) we need only establish (iii), since the only possible way β might not be infinite is if some α receives attention infinitely often for $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$. Thus suppose $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$. Now the only way α might need attention infinitely often is if its follower keeps getting cancelled. Cancellation of such a follower y only happens if, for some $\hat{\gamma} \subset \gamma$ with $\mathrm{lh}(\hat{\gamma}) \equiv 2 \pmod 3$, at the close of a $\hat{\gamma}$-gap we have seen $\gamma^p(x, \hat{\gamma}, s)$ ($p \in \{0, 2\}$) or $\rho(x, \hat{\gamma}, s)$ need changing, for $x \leq y$. By choice of s_1 and initialization of α when some $\hat{\alpha} \subset \alpha$ receives attention we might as well suppose that α has no follower at stage s_1. Furthermore we may suppose (by (iv)) a stage s_3 such that for every link (τ, η) with $\eta \subset \alpha$ or $\tau \subset \alpha$ that existed at stage s_1, there has been a stage $s_2 = s_2(\tau, \eta) < s_3$ where (τ, η) has been removed. By cancellation we can suppose that α has no follower at stage s_3. Now at the least genuine α-stage after s_3, α gets a follower $y = y(\alpha, s_3)$.

We claim that y cannot be cancelled. The only way y could be cancelled is if for some $\hat{\gamma}\hat{\ }g \subset \alpha$ and follower $x < y$ we have seen $\gamma^p(x, \hat{\gamma}, s_3)$ or $\rho(x, \hat{\gamma}, s_3)$ need changing at the stage $s_4 > s_3$ where the $\hat{\gamma}$-gap is closed. However by choice of s_3 and induction the only followers $z < s_3$ left alive and targeted for A at stage s_3 are followers of $\hat{\alpha} \subsetneq \alpha$, which will *never* enter A.

It follows that at stage s_4, case 3, subcase 6, case 2(b) pertains to x and $\hat{\gamma}$. This case specifically protects y from cancellation since no x-computations have changed since stage s_3. This gives the claim. Therefore α can receive attention at most once more (meeting $P_{e(\alpha)}$) and this gives the lemma. □

(3.13) **Lemma.** *Let $\alpha \subset \beta$ with $\mathrm{lh}(\alpha) \equiv 1 \pmod 3$ and let y be a follower of α where $y = \lim_s y(\alpha, s)$ (i.e. the last follower of α appointed at stage s_1, say,* *assuming* $W_{e,s_1} \cap A_{s_1} = \emptyset$). *Then*

(i) $(\forall s \in G^{\alpha})[s > s_1 \to y > \hat{r}(\alpha, s)]$ *where* $\hat{r}(\alpha, s) = \max\{r(\gamma, s) : \gamma \leqslant_{\mathrm{L}} \alpha\}$, *and*

(ii) *hence* $P_{e(\alpha)}$ *is met.*

Proof. At stage s_1, $y = \langle \alpha, s_1 \rangle$ and so y exceeds all $\hat{r}(\alpha, s_1)$ by convention. It suffices to claim that $\hat{r}(\alpha, s_1) = \hat{r}(\alpha, s)$ for all $s > s_1$ with $s \in G^{\alpha}$. Indeed we claim that for all $\gamma \leqslant_{\mathrm{L}} \alpha$, $r(\gamma, s_1) = r(\gamma, s)$ for such s. If $r(\gamma, s_1) \neq r(\gamma, s)$, then it can only be that $\gamma\hat{\ }w$ or $\gamma\hat{\ }f \subseteq \alpha$. In the first case, $r(\gamma, s)$ would only be reset after a $\gamma\hat{\ }g$-stage $\hat{s}$ and this would cancel y since $\hat{s} > s_1$ and $\gamma\hat{\ }g <_{\mathrm{L}} \alpha$ with $\gamma\hat{\ }g \nsubseteq \alpha$. In the second case if $r(\gamma, s) \neq r(\gamma, s_1)$ it must be that γ receives attention at stage s. Thus in this case too we cancel y. Thus these cases can't occur and so $r(\gamma, s_1) = r(\gamma, s)$. This gives (i) and (ii) follows by the usual argument. $\square$

(3.14) **Lemma.** *Suppose* $\alpha \subset \beta$ *with* $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ *and* $\alpha\hat{\ }1 \subset \beta$. *Then one of the following holds for some* $j \in \{0, 1, 2\}$ *or* $k \in \{0, 2\}$:

$$\Lambda^{j}_{e(\alpha)}(A) \neq W^{j}_{e(\alpha)} \quad \textit{or} \quad \Phi^{k}_{e(\alpha)}(W^{k+1}_{e(\alpha)} \oplus W^{k+2}_{e(\alpha)}) \neq W^{k}_{e(\alpha)}.$$

Proof. $\mathrm{Lim}\{l(e(\alpha), s) : s \in G^{\alpha}\} < \infty$, hence one of the inequalities above must hold. $\square$

Similarly we see that

(3.15) **Lemma.** *Suppose that* $\alpha \subset \beta$ *with* $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$ *and* $\alpha\hat{\ }w \subset \beta$. *Then* $\Psi_{i(\alpha)}(W^{1}_{e(\alpha)}) \neq Q_{\tau(\alpha)}$.

Also we need that

(3.16) **Lemma.** *Suppose* $\alpha \subset \beta$ *with* $\mathrm{lh}(\alpha) \equiv 0 \pmod 3$ *and* $\alpha\hat{\ }0 \subset \beta$. *Then* $Q_{\alpha} \leqslant_{\mathrm{T}} W^{0}_{e(\alpha)}, W^{2}_{e(\alpha)}$.

Proof. Q_{α} is the collection of $x(\gamma, i, s) = x(\gamma)$ enumerated at the successful closure of γ-gap where $\alpha = \tau(\gamma)$. Note that all γ-gaps opened are eventually closed or cancelled. This follows by (3.12). At the successful closure, we must have $W^{p}_{e(\alpha)}[\gamma^{p}(x, \alpha, s)]$ changing for $p = 0, 2$ and furthermore γ^{p} is reset only if $W^{p}_{e(\alpha)}$ changes.

This means that it suffices to show that $\gamma^{p}(x, \alpha, s)$ can be reset at most finitely often for $p \in \{0, 2\}$. However $\gamma^{p}(x, \alpha, s)$ is only reset at the unsuccessful closure of an α-gap, and case 2 of subcase 6 pertains. When this pertains at stage s, say, we cancel all followers $y > x$ targeted for A and raise $r(\alpha, s) = s$ until the next gap. Such changes are therefore caused only by numbers $<x$ entering A. As we discussed in the basic module: since the restraint protects these computations, and followers appointed after stage s must exceed s (see (3.2)), for this point it follows that γ^{p} can be reset at most x times, giving the lemma. $\square$

(3.17) **Lemma** (Truth of outcome). *Suppose* $\alpha \subset \beta$ *with* $\mathrm{lh}(\alpha) \equiv 2 \pmod 3$ *and* $\alpha^\wedge a \subset \beta$.

(i) *If* $a = f$, *then* $\Psi_{i(\alpha)}(W^1_{e(\alpha)}) \neq Q_{\tau(\alpha)}$.

(ii) *If* $a = g$, *then* $W^0_{e(\alpha)} \leq_{\mathrm{T}} W^1_{e(\alpha)}$.

Proof. (i) Let s_1 be a stage as in (3.12). Then if $a = f$ some follower x_k is enumerated into $Q_{\tau(\alpha)}$ at some stage s to create a disagreement and $r(\alpha, s) = s$. By choice of s_1 this restraint is not violated.

(ii) Suppose that $a = g$ and let s_1 be the relevant stage which is good for α as in (3.12). We show how to compute arbitrarily long initial segments of $W^0_{e(\alpha)}$. Note that choice of s_1 means that after stage s_1, α cannot be initialized. Moreover, at the close of every unsuccessful α-gap, α gets a new follower hence $x_k = \lim_s x(\alpha, k, s)$ exists. Indeed, if $s > s_1$ and $x(\alpha, k, s)$ is defined at stage s, then $x(\alpha, k, s) = x_k$. Now we reason virtually exactly as for the basic module and we simply sketch to compute $W^0_{e(\alpha)}[z]$. Find a stage $s_2 > s_1$ where $x = x_k > z$ is appointed to α. At this stage we create a link (τ, α) where $\tau = \tau(\alpha)$. When this link at $s_3 > s_2$ is removed, we cancel all followers m targeted at A with z and define $\gamma^p(x, s_3) = \gamma^p(x, \alpha, s_3)$ for $p \in \{0, 2\}$, and set $r(\alpha, s_3) = s_3$. These restraints are not violated by choice of s_3 until we open an α-gap (where we define $\rho(x, \alpha, s_3)$). Since this gap is closed unsuccessfully at some α-stage $s_4 > s_3$, and so case 3, subcase 6 pertains. If case 1 of this subcase holds, then nothing has changed and all the γ^p, ρ-computations are the same as they were at stage s_3. In any of the other cases of subcase 6 (except case 2(e)) we get to reset ρ but cancel all followers $y > x$ targeted for A. In case 2(e) we only lose the outer layer but again get to cancel. Again the cancellation procedure ensures that ρ gets reset at most x times. Of course to decide what ρ is reset to we need only go to the next α-stage exceeding s_4. The whole point is that like the basic module $W^0_e[x]$ can only change between expansionary stages if $W^1_e[\rho(x, s)]$ does. Hence $W^0_e \leq_{\mathrm{T}} W^1_e$. □

(3.18) **Variations and comments.** What is really important here is closing gaps at α-stages and 'confirming' followers to define γ^0, γ^2 and ρ. With a little more care, one can combine the argument above with permitting to show:

(3.19) *Every nonzero r.e. degree has a predecessor that bounds no critical triple.*

The reader may note that what we also ensure in our construction is that for all x, for all functionals Φ and r.e. sets $W \leq_{\mathrm{T}} A$ there is a recursive enumeration of W and Φ such that if $\Phi(W)$ is r.e. then

$$\{s : W_s[u(\Phi_s(W_s; x))] \neq W_{s+1}[u(\Phi_s(W_s; x))]\} \leq x.$$

It is unclear if some condition like this on A is sufficient. It would seem to be related to the Mohrherr/Bickford–Mills *superlow degrees* (see e.g. Mohrherr [11]) and the *array recursive degrees* of Downey–Jockusch–Stob [5].

We remark here though that it is possible to use the techniques of [5] to create an array nonrecursive (see [5] for the definition) degree **a** that bounds no critical triple. We also remark that the degree we construct seems to be *Wtt-topped* in the sense that for all r.e. $B \leqslant_T A$, $B \leqslant_{Wtt} A$. Thus perhaps the result is really related to the Wtt-structure inside deg(A).

I cannot resist at this point including a very clever observation of Ambos-Spies to show that the results of Section 3 cannot be improved to construct distributive initial segments of **R**. This is because N_5 is dense in **R**. Here is Ambos-Spies' argument:

Let $\mathbf{a} < \mathbf{b}$. By Slaman's density theorem there exists $\mathbf{c} \mid \mathbf{d}$ with $\mathbf{a} < \mathbf{c}$, $\mathbf{d} < \mathbf{b}$ with $\mathbf{e} = \mathbf{c} \cap \mathbf{d}$. Now as the nonbranching degrees are dense (Fejer [6]), there exists a nonbranching degree $\mathbf{f}$ with $\mathbf{e} < \mathbf{f} \leqslant \mathbf{c}$. Let $\mathbf{g} = \mathbf{f} \cup \mathbf{d}$. Now as $\mathbf{f}$ is nonbranching and $\mathbf{c}$, $\mathbf{f} \cup \mathbf{d} > \mathbf{f}$ there exists a degree $\mathbf{h}$ with $\mathbf{f} < \mathbf{h} < \mathbf{c}$, $\mathbf{f} \cup \mathbf{d}$. Since $\mathbf{h} < \mathbf{c}$ we know that $\mathbf{h} \cap \mathbf{d} = \mathbf{e}$. As $\mathbf{f} \leqslant \mathbf{h} \leqslant \mathbf{f} \cup \mathbf{d}$ we know that $\mathbf{h} \cup \mathbf{d} = \mathbf{f} \cup \mathbf{d}$. Therefore the degrees $\mathbf{e}$, $\mathbf{h}$, $\mathbf{f}$, $\mathbf{d}$, $\mathbf{f} \cup \mathbf{d}$ give the embedding of N_5.

Remark. The technical difference between, say, M_5 and N_5 above is that M_5 seems to need continuous appointment of traces whereas N_5 only needs a bounded number. This is why we can't use the 'layering' technique of Section 3 to kill N_5.

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